Lagrangian description of nonlinear dust–ion acoustic waves in dusty plasmas

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Abstract. An analytical model is presented for the description of nonlinear dust-ion-acoustic waves propagating in an unmagnetized, collisionless, three component plasma composed of electrons, ions and inertial dust grains. The formulation relies on a Lagrangian approach of the plasma fluid model. The modulational stability of the wave amplitude is investigated. Different types of localized envelope electrostatic excitations are shown to exist.

1 Introduction

In the last two decades, dusty plasmas (DP) have attracted a great deal of attention due to a variety of new phenomena observed in them and the novel physical mechanisms involved in their description [1,2]. In addition to known plasma electrostatic modes [3], new oscillatory modes arise in DP [1,2], among which the *dust-ion acoustic wave* (DIAW) and *dust acoustic waves* (DAW) are of significant interest in laboratory dusty plasma discharges. In the DIAW the restoring force comes from the pressures of inertialess electrons, whereas the ion mass provides the inertia, similar to the usual ion-acoustic waves in an electron-ion plasma. Thus, the DIAW is characterized by a phase speed much smaller (larger) than the ion (electron) thermal speed, and a frequency much higher than the dust plasma frequency $\omega_{p,d}$; therefore, on the timescale of our interest, stationary dust grains do not participate in the wave dynamics; they just affect the equilibrium quasi-neutrality condition. As a matter of fact, the DIAW phase velocity is higher than that of IA waves, due to the electron density depletion in the background plasma when dust grains are negatively charged; quite remarkably, this fact results in suppression of the Landau damping mechanism [1], known to prevail over the IAW propagation in an electron-ion plasma [3].

The linear properties of the IAWs have been quite extensively studied and now appear well understood [1]. As far as nonlinear effects are concerned, various studies have pointed out the possibility of the formation of DIAW-related localized structures, due to a mutual compensation between nonlinearity and dispersion, including small-amplitude pulse solitons, shocks and vortices [4]. Furthermore, the propagation of nonlinearly modulated DIA wave packets was studied in reference [5], in addition to the formation of localized envelope soliton-modulated waves due to the modulational instability of the carrier waves. A very interesting known approach, not yet included in our current knowledge with respect to the DIA plasma waves, is the Lagrangian description of a nonlinear wave profile. In the context of electrostatic plasma waves, this formalism has been employed in studies of electron plasma waves [6–8] and, more recently, ion-acoustic [9] and dust-acoustic [10] waves. Our aim here is to extend previous results by applying the Lagrangian formalism to the description of nonlinear DIAWs propagating in dusty plasmas.

We shall consider the nonlinear propagation of dust-ion-acoustic waves in a collisionless plasma consisting of three distinct particle species 'α': an inertial species of ions (denoted by 'i'; mass m_i, charge $q_i = +Z_i e$; e denotes the absolute of the electron charge), surrounded by an environment of thermalized electrons (mass m_e, charge $-e$), and massive dust grains (mass M, charge $q_d = sZ_d e$, both assumed constant for simplicity); Z_d denotes the charge state of dust grains; we leave the choice of dust grain charge sign $s = q_d/|q_d|$ ($= -1/+1$ for negative/positive dust charge) open in the algebra. Charge neutrality is assumed at equilibrium.

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2 The model

Let us consider the hydrodynamic–Poisson system of equations which describe the evolution of the ion "fluid" in the plasma. The ion number density n_i is governed by the continuity equation

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \, \mathbf{u}_i) = 0, \tag{1}$$

where the mean velocity $\mathbf{u}_i$ obeys

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = \frac{Z_i e}{m_i}\mathbf{E} = -\frac{Z_i e}{m_i} \nabla \, \Phi. \tag{2}$$

The electric field $\mathbf{E} = -\nabla \Phi$ is related to the gradient of the wave potential Φ, which is obtained from Poisson's equation $\nabla \cdot \mathbf{E} = 4\pi \sum q_s \, n_s$, viz.

$$\nabla^2 \Phi = 4\pi \, e \, (n_i + n_e - Z_i \, n_i). \tag{3}$$

Alternatively, one may consider

$$\frac{\partial E}{\partial t} = -4\pi \sum_\alpha q_\alpha \, n_\alpha \, u_\alpha. \tag{4}$$

We assume a near-Boltzmann distribution for the electrons, i.e. $n_e \approx n_{e,0} \exp(e\Phi/k_B T_e)$ (T_e is the electron temperature and k_B is Boltzmann's constant). The dust distribution is assumed stationary, i.e. $n_d \approx$ const. The overall quasi-neutrality condition at equilibrium then reads

$$Z_i \, n_{i,0} + s Z_d \, n_d - n_{e,0} = 0. \tag{5}$$

2.1 Reduced Eulerian equations

By choosing appropriate physical scales, equations (1–3) can be cast into a reduced (dimensionless) form. Let us define the ion-acoustic speed $c_s = (k_B T_e/m_i)^{1/2}$. An appropriate choice for the space and timescales, L and $T = L/c_s$, are the effective Debye length $\lambda_D = (k_B T_e/4\pi Z_i^2 n_{i,0} e^2)^{1/2} \equiv c_s/\omega_{p,i}$ and the ion plasma period $\omega_{p,i}^{-1} = (4\pi n_{i,0} Z_i^2 e^2/m_i)^{-1/2}$, respectively. Alternatively, one might leave the choice of L (and thus $T = L/c_s$) arbitrary — following an idea suggested in references [9, 10] — which leads to the appearance of a dimensionless dispersion parameter $\delta = 1/(\omega_{p,i} T) = \lambda_D/L$ in the formulae. The specific choice of scale made above corresponds to $\delta = 1$ (implied everywhere in the following, unless otherwise stated); however, we may keep the parameter δ to 'label' the dispersion term in the forthcoming formulae.

For one-dimensional wave propagation along the x-axis, equations (1–3) can now be written as

$$\frac{\partial n}{\partial t} + \frac{\partial (n \, u)}{\partial x} = 0,$$
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\nabla\phi,$$
$$\delta^2 \frac{\partial^2 \phi}{\partial x^2} = \hat{n} - n, \tag{6}$$

where all quantities are dimensionless: $n = n_i/n_{i,0}$, $\mathbf{u} = \mathbf{u}_i/v_0$ and $\phi = \Phi/\Phi_0$; the scaling quantities are, respectively: the equilibrium ion density $n_{i,0}$, the effective sound speed $v_0 = c_s$ (defined above) and $\Phi_0 = k_B T_e/(Z_i e)$. The (reduced) electron and dust background density $\hat{n}$ is defined as

$$\hat{n} = \frac{n_e}{Z_i n_{i,0}} \, e^{\phi/Z_i} + s \frac{Z_d n_d}{Z_i n_{i,0}} \equiv \mu \, e^{\phi/Z_i} + 1 - \mu, \tag{7}$$

where we have defined the DP parameter $\mu = n_{e,0}/(Z_i n_{i,0})$, and made use of equation (5). Note that both n and $\hat{n}$ reduce to unity at equilibrium.

We shall define, for later reference, the function $f(\phi) = \hat{n}$ — given by equation (7) — and its inverse function

$$f^{-1}(x) = Z_i \, \ln\left(1 + \frac{x-1}{\mu}\right) \equiv g(x), \tag{8}$$

viz. $f(\phi) = x$ implies $\phi = f^{-1}(x) \equiv g(x)$.

We note that the dependence on the charge sign s is now incorporated in $\mu = 1 + s Z_d n_{d,0}/(Z_i n_{i,0})$; retain that $\mu < 1$ ($\mu > 1$) corresponds to negative (positive) dust. Remarkably, since the dust-free limit is recovered for $\mu = 1$, the results to be obtained in the following are also straightforward valid for ion-acoustic waves propagating in (dust-free) e-i plasma, upon setting $\mu = 1$ in the formulae.

The well–known DIAW dispersion relation $\omega^2 = c_s^2 k^2/(k^2 \lambda_D^2 + 1)$ [5] is obtained from equations (1–5). On the other hand, the system (6) yields the reduced relation $\omega^2 = k^2/(\delta^2 k^2 + 1)$, which of course immediately recovers the former dispersion relation upon restoring dimensions (regardless, in fact, of one's choice of space scale L; cf. definition of δ). However, some extra qualitative information is admittedly hidden in the latter (dimensionless) relation. Should one consider a very long space scale $L \gg \lambda_D$ (i.e. $\delta \ll 1$), one readily obtains $\omega \sim k$ (unveiling the role of δ as a characteristic dispersion control parameter). Finally, the opposite limit of short L (or infinite δ) corresponds to ion plasma oscillations (viz. $\omega = \omega_{p,i} = $ const).

2.2 Lagrangian description

Let us introduce the Lagrangian variables $\{\xi, \tau\}$, which are related to the Eulerian ones $\{x, t\}$ via

$$\xi = x - \int_0^\tau u(\xi, \tau') \, d\tau', \qquad \tau = t. \tag{9}$$

See that they coincide at $t = 0$. Accordingly, the space and time gradients are transformed as

$$\partial/\partial x \to \alpha^{-1} \, \partial/\partial \xi, \qquad \partial/\partial t \to \partial/\partial \tau - \alpha^{-1} u \, \partial/\partial \xi,$$

where we have defined the quantity

$$\alpha(\xi, \tau) \equiv \frac{\partial x}{\partial \xi} = 1 + \int_0^\tau d\tau' \frac{\partial}{\partial \xi} u(\xi, \tau'). \tag{10}$$

Note that the convective derivative $D \equiv \partial/\partial t + u\,\partial/\partial x$ is now plainly identified to $\partial/\partial\tau$. Also notice that α satisfies $\alpha(\xi, \tau = 0) = 0$ and

$$\frac{\partial\alpha(\xi,\tau)}{\partial\tau} = \frac{\partial u(\xi,\tau)}{\partial\xi}. \tag{11}$$

As a matter of fact, the Lagrangian transformation defined here reduces to a Galilean transformation if one suppresses the evolution of u, i.e. for $u = \text{const}$ (or $\partial u/\partial\tau = \partial u/\partial\xi = 0$, hence $\alpha = 1$). Furthermore, if one also suppresses the dependence in time τ, this transformation is reminiscent of the travelling wave ansatz $f(x,t) = f(x - vt \equiv s)$, which is widely used in the Sagdeev potential formalism [4,11].

The Lagrangian variable transformation defined above leads to a new set of reduced equations

$$n(\xi,\tau) = \alpha^{-1}(\xi,\tau)\,n(\xi,0) \tag{12}$$

$$\frac{\partial u(\xi,\tau)}{\partial\tau} = \frac{Z_i e}{m_i} E(\xi,\tau)$$
$$= -\frac{Z_i e}{m_i}\,\alpha^{-1}(\xi,\tau)\,\frac{\partial\phi(\xi,\tau)}{\partial\xi} \tag{13}$$

$$\alpha^{-1}(\xi,\tau)\,\frac{\partial E(\xi,\tau)}{\partial\xi} = 4\pi Z_i e[n(\xi,\tau) - \hat{n}\,n_{i,0}] \tag{14}$$

$$\left(\frac{\partial}{\partial\tau} - \alpha^{-1}u\,\frac{\partial}{\partial\xi}\right)E(\xi,\tau) = -4\pi Z_i e n(\xi,\tau)u(\xi,\tau), \tag{15}$$

where we have temporarily restored dimensions for physical transparency; recall that the (dimensionless) quantity $\hat{n}$, which is in fact a function of ϕ, is given by (7). One immediately recognizes the role of the (inverse of the) function $\alpha(\xi,\tau)$ as a density time evolution operator; cf. equation (12) [12]. Poisson's equation is now obtained by eliminating ϕ from equations (13, 14)

$$\alpha^{-1}\frac{\partial}{\partial\xi}\left(\alpha^{-1}\frac{\partial\phi}{\partial\xi}\right) = -4\pi Z_i e(n - \hat{n}\,n_{i,0}). \tag{16}$$

Note that a factor δ^2 should appear in the left-hand side if one rescaled equation (16) as described above; cf. the last of equations (6). This will be retained for later reference, with respect to the treatment suggested in reference [9] (see discussion below).

In principle, our aim is to solve the system of equations (12–15) or, by eliminating ϕ, equations (12, 13, 16) for a given initial condition $n(\xi, \tau = 0) = n_0(\xi)$, and then make use of the definition (9) in order to invert back to the Eulerian arguments of the state moment variables (i.e. density, velocity etc.). However, this abstract scheme is definitely not a trivial task to accomplish.

3 Nonlinear dust-ion acoustic oscillations

Multiplying equation (14) by $u(\xi,\tau)$ and then adding to equation (15), one obtains

$$\frac{\partial E(\xi,\tau)}{\partial\tau} = -4\pi Z_i e n_{i,0}\,\hat{n}\,u(\xi,\tau). \tag{17}$$

Combining with equation (13), one obtains

$$\frac{\partial^2 u}{\partial\tau^2} = -\omega_{p,i}^2\,\hat{n}\,u, \tag{18}$$

where $\omega_{p,i}$ is the ion plasma frequency (defined above). Despite its apparent simplicity, equation (18) is *neither* an ordinary differential equation (ODE) — since all variables depend on *both* time τ and space ξ — nor a closed evolution equation for the mean velocity $u(\xi,\tau)$: note that the (normalized) background particle density $\hat{n}$ depends on the potential ϕ and on the plasma parameters; see its definition (7). The evolution of the potential $\phi(\xi,\tau)$, in turn, involves $u(\xi,\tau)$ (via the quantity $\alpha(\xi,\tau)$) and the ion density $n(\xi,\tau)$.

Equation (18) suggests that the system performs nonlinear oscillations at a frequency $\omega = \omega_{p,i}\,\hat{n}^{1/2}$. Near equilibrium, the quantity $\hat{n}$ is approximately equal to unity and one plainly recovers a linear oscillation at the ion plasma frequency $\omega_{p,i}$. Quite unfortunately this apparent simplicity, which might in principle enable one to solve for $u(\xi,\tau)$ and then obtain $\{\xi,\tau\}$ in terms of $\{x,t\}$ and vice versa (cf. Davidson's treatment for electron plasma oscillations in Ref. [7]; also compare to Ref. [8], setting $\gamma = 0$ therein), is absent in the general (off-equilibrium) case where the plasma oscillations described by equation (18) are intrinsically *nonlinear*.

Since equation (18) is in general not a closed equation for u, unless the background density $\hat{n}$ is constant (i.e. independent of ϕ, as in Refs. [7,8]), one can *neither* apply standard methods involved in the description of nonlinear oscillators on equation (18) (cf. Ref. [8]), *nor* reduce the description to a study of equations (17, 18) (cf. Ref. [6]), but rather has to retain all (or rather five) of the evolution equations derived above, since five inter-dependent dynamical state variables (i.e. n, u, E, ϕ and α) are involved. This procedure will be exposed in the following section.

4 Perturbative nonlinear Lagrangian treatment

Let us consider weakly nonlinear oscillations performed by our system close to (but not at) equilibrium. The basis of our study will be the reduced system of equations

$$\frac{\partial}{\partial\tau}(\alpha\,n) = 0,$$
$$\frac{\partial u}{\partial\tau} = E,$$
$$\frac{\partial E}{\partial\xi} = (n - \hat{n})\,\alpha,$$
$$\alpha E = -\frac{\partial\phi}{\partial\xi},$$
$$\frac{\partial\alpha}{\partial\tau} = \frac{\partial u}{\partial\xi}, \tag{19}$$

which follow from the Lagrangian equations (12–16) by scaling over appropriate quantities, as described in Section 2.1 [13]. This system describes the evolution of

the state vector, say $\mathbf{S} = (\alpha, n, u, E, \phi)$ ($\in \Re^5$), in the Lagrangian coordinates defined above. We will consider small deviations from the equilibrium state $\mathbf{S}_0 = (1, 1, 0, 0, 0)^T$, by taking $\mathbf{S} = \mathbf{S}^{(0)} + \epsilon \mathbf{S}_1^{(0)} + \epsilon^2 \mathbf{S}_2^{(0)} + ...$, where ϵ ($\ll 1$) is a smallness parameter. Accordingly, we shall Taylor develop the quantity $\hat{n}(\phi)$ near $\phi \approx 0$, viz. $\phi \approx \epsilon \phi_1 + \epsilon^2 \phi_2 + ...$, in order to express $\hat{n}$ as

$$\begin{aligned}
\hat{n} &\approx 1 + c_1 \phi + c_2 \phi^2 + c_3 \phi^3 + ... \\
&= 1 + \epsilon c_1 \phi_1 + \epsilon^2 (c_1 \phi_2 + c_2 \phi_1^2) \\
&\quad + \epsilon^3 (c_1 \phi_3 + 2 c_2 \phi_1 \phi_2 + c_3 \phi_1^3) + ...,
\end{aligned} \qquad (20)$$

where the coefficients c_j ($j = 1, 2, ...$), which are determined from the definition (7) of $\hat{n}$, contain all the essential dependence on the plasma parameters, e.g. μ; making use of $e^x \approx \sum_{n=0}^{\infty} x^n / n!$, one readily obtains

$$c_1 = \mu/Z_i, \qquad c_2 = \mu/(2Z_i^2), \qquad c_2 = \mu/(6Z_i^3).$$

Remember that for $\mu = 1$ (i.e. for vanishing dust) one recovers the expressions for IAWs in e-i plasma.

Following the standard reductive perturbation technique [14], we shall consider the stretched (slow) Lagrangian coordinates $Z = \epsilon(\xi - \lambda\tau)$, $T = \epsilon^2 \tau$ (where $\lambda \in \Re$ will be determined later). The perturbed state of (the jth — $j = 1, ..., 5$ — component of) the state vector $\mathbf{S}^{(n)}$ is assumed to depend on the fast scales via the carrier phase $\theta = k\xi - \omega\tau$, while the slow scales enter the argument of the (jth element's) lth harmonic amplitude $S_{j,l}^{(n)}$, viz. $Sj^{(n)} = \sum_{l=-\infty}^{\infty} S_{j,l}^{(n)}(Z, T) e^{il(k\xi - \omega\tau)}$ (where $S_{j,-l}^{(n)} = S_{j,l}^{(n)*}$ ensures reality). Treating the derivative operators as

$$\frac{\partial}{\partial \tau} \rightarrow \frac{\partial}{\partial \tau} - \epsilon\lambda\frac{\partial}{\partial Z} + \epsilon^2 \frac{\partial}{\partial T}, \qquad \frac{\partial}{\partial \xi} \rightarrow \frac{\partial}{\partial \xi} + \epsilon\frac{\partial}{\partial Z},$$

and substituting into the system of evolution equations, one obtains an infinite series in both (perturbation order) ϵ^n and (phase harmonic) l. The standard perturbation procedure now consists in solving in successive orders $\sim \epsilon^n$ and substituting in subsequent orders. The method involves a tedious calculation which is however straightforward; the details of the method are presented e.g. in reference [5], so only the essential stepstones need to be provided here.

The equations obtained for $n = l = 1$ determine the first harmonics of the perturbation

$$\begin{aligned}
n_1^{(1)} &= -\alpha_1^{(1)} = (k^2/\omega^2)\psi, \\
u_1^{(1)} &= (k/\omega)\psi, \qquad E_1^{(1)} = -ik\psi
\end{aligned} \qquad (21)$$

where ψ denotes the potential correction $\phi_1^{(1)}$. The cyclic frequency ω obeys the dispersion relation $\omega^2 = k^2/(k^2 + sc_1)$, which exactly recovers, once dimensions are restored, the standard IAW dispersion relation [3] mentioned above.

Proceeding in the same manner, we obtain the second order quantities, namely the amplitudes of the second harmonics $\mathbf{S}_2^{(2)}$ and constant ('direct current') terms $\mathbf{S}_0^{(2)}$, as

well as a finite contribution $\mathbf{S}_1^{(2)}$ to the first harmonics; as expected from similar studies, these three (sets of 5, at each n, l) quantities are found to be proportional to ψ^2, $|\psi|^2$ and $\partial\psi/\partial Z$ respectively; the lengthy expressions are omitted here for brevity. The ($n = 2$, $l = 1$) equations provide the compatibility condition: $\lambda = \omega(1 - \omega^2)/k = d\omega/dk$; λ is therefore the group velocity $v_g(k) = \omega'(k)$ at which the wave envelope propagates. It turns out that v_g decreases with increasing wave number k; nevertheless, it always remains positive.

In order $\sim \epsilon^3$, the equations for $l = 1$ yield an explicit compatibility condition in the form of a nonlinear Schrödinger–type equation (NLSE)

$$i\frac{\partial\psi}{\partial T} + P\frac{\partial^2\psi}{\partial Z^2} + Q|\psi|^2\psi = 0. \qquad (22)$$

Recall that $\psi \equiv \phi_1^{(1)}$ denotes the amplitude of the first-order electric potential perturbation. The 'slow' variables $\{Z, T\}$ were defined above.

The *dispersion coefficient* P is related to the curvature of the dispersion curve as $P = \omega''(k)/2 = -3\omega^3(1 - \omega^2)/(2k^2)$. One may easily check that P is negative (for all values of k).

The *nonlinearity coefficient* Q is due to carrier wave self-interaction. It is given by the expression

$$\begin{aligned}
Q = &+\frac{\omega^3}{12\,k^4}\frac{\mu}{Z_i^4}\Big[3Z_i^3k^6 - 3(\mu + 4)Z_i^2k^4 \\
&+ 3(1 - 2\mu - 5\mu^2)Z_ik^2 - \mu(3\mu - 1)^2\Big].
\end{aligned} \qquad (23)$$

where the coefficients $c_{1,2,3}$ were defined above.

For low wavenumber k, Q goes to $-\infty$ as

$$Q \approx -\frac{(3\mu - 1)^2\mu^{1/2}}{12\,Z_i^{5/2}}\frac{1}{k}.$$

4.1 Modulational stability analysis

According to the standard analysis [15], we can linearize around the plane wave solution of the NLSE (22) $\psi = \hat{\psi}\,e^{iQ|\hat{\psi}|^2\tau} + c.c.$, (c.c.: complex conjugate) — notice the amplitude dependence of the frequency shift $\Delta\omega = \epsilon^2 Q|\hat{\psi}|^2$ — by setting $\hat{\psi} = \hat{\psi}_0 + \epsilon\,\hat{\psi}_1$, and then assuming the perturbation $\hat{\psi}_1$ to be of the form: $\hat{\psi}_1 = \hat{\psi}_{1,0}\,e^{i(\hat{k}\zeta - \hat{\omega}\tau)} + c.c.$ Substituting into (22), one thus readily obtains $\hat{\omega}^2 = P^2\hat{k}^2(\hat{k}^2 - 2(Q/P)|\hat{\psi}_{1,0}|^2)$. The wave will thus be *stable* ($\forall\hat{k}$) if the product PQ is negative. However, for positive $PQ > 0$, instability sets in for wavenumbers below a critical value $\hat{k}_{cr} = \sqrt{2Q/P}\,|\hat{\psi}_{1,0}|$, i.e. for wavelengths above a threshold $\lambda_{cr} = 2\pi/\hat{k}_{cr}$; defining the instability growth rate $\sigma = |\text{Im}\,\hat{\omega}(\hat{k})|$, we see that it reaches its maximum value for $\hat{k} = \hat{k}_{cr}/\sqrt{2}$, viz.

$$\sigma_{max} = |\text{Im}\,\hat{\omega}|_{\hat{k}=\hat{k}_{cr}/\sqrt{2}} = |Q|\,|\hat{\psi}_{1,0}|^2.$$

We see that the instability condition depends only on the sign of the product PQ, which may be studied numerically, relying on the expressions derived above.

4.2 Finite amplitude nonlinear excitations

The NLSE (22) is long known to possess distinct types of localized constant profile (solitary wave) solutions, depending on the sign of the product PQ [5,15,16]. Remember that this equation here describes the evolution of the wave's envelope, so these solutions represent slowly varying localized envelope structures, confining the (fast) carrier wave. The analytic form of these excitation can be found in the literature (see e.g. in [5] for a brief review) and need not be derived here in detail. Let us however briefly summarize those results.

Following reference [16], we may seek a solution of equation (22) in the form $\psi(\zeta, \tau) = \rho(Z, T)\, e^{i\,\Theta(\zeta,\tau)} + c.c.$, where ρ, σ are real variables which are determined by substituting into the NLSE and separating real and imaginary parts. The different types of solution thus obtained are summarized in the following.

For $PQ > 0$ we find the *(bright) envelope soliton*

$$\rho = \pm \rho_0 \,\mathrm{sech}\!\left(\frac{Z - u_e\,\tau}{L}\right),$$
$$\Theta = \frac{1}{2P}\left[u_e Z - \left(\Omega + \frac{1}{2}u_e^2\right)T\right], \qquad (24)$$

which represents a localized pulse travelling at the envelope speed u_e and oscillating at a frequency Ω (at rest). The pulse width L depends on the maximum amplitude square ρ_0 as $L = (2P/Q)^{1/2}/\rho_0$. Since the product PQ is always positive for long wavelengths, as we saw above, this type of excitation will be rather privileged in dusty plasmas. The bright-type envelope soliton is depicted in Figures 1a and 1b.

For $PQ < 0$, we obtain the *dark* envelope soliton (*hole*) [16]

$$\rho = \pm \rho_1 \left[1 - \mathrm{sech}^2\!\left(\frac{Z - u_e T}{L'}\right)\right]^{1/2}$$
$$= \pm \rho_1 \,\tanh\!\left(\frac{Z - u_e T}{L'}\right),$$
$$\Theta = \frac{1}{2P}\left[u_e Z - \left(\frac{1}{2}u_e^2 - 2PQ\rho_1\right)T\right], \qquad (25)$$

which represents a localized region of negative wave density (shock) travelling at a speed u_e; see Figure 1c. Again, the pulse width depends on the maximum amplitude square ρ_1 via $L' = (2\lvert P/Q\rvert)^{1/2}/\rho_1$.

Finally, still for $PQ < 0$, one also obtains the *gray* envelope solitary wave [16]

$$\rho = \pm \rho_2 \left[1 - a^2 \,\mathrm{sech}^2\!\left(\frac{Z - u_e T}{L''}\right)\right]^{1/2}, \qquad (26)$$
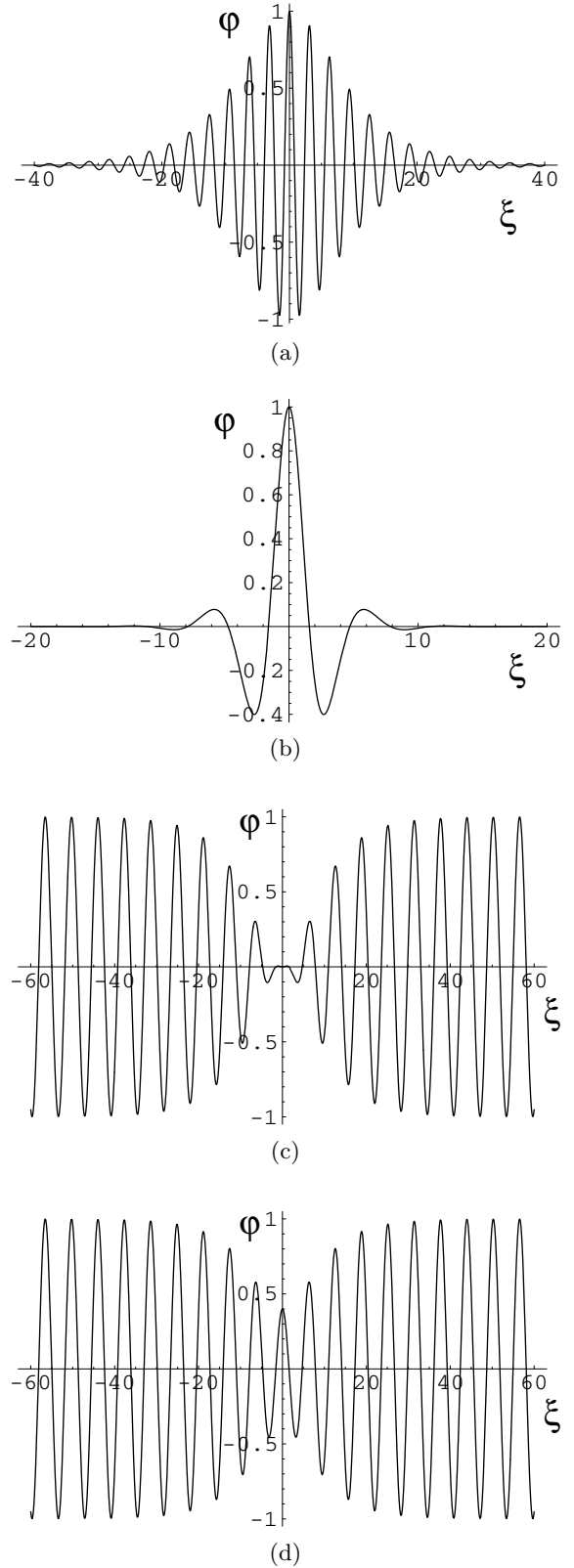

Fig. 1. A heuristic representation of wave packets modulated by solutions of the NLS equation. These envelope excitations are of the: (a, b) bright type ($PQ > 0$, pulses); (c) dark type, (d) gray type ($PQ < 0$, voids). Notice that the amplitude never reaches zero in (d).
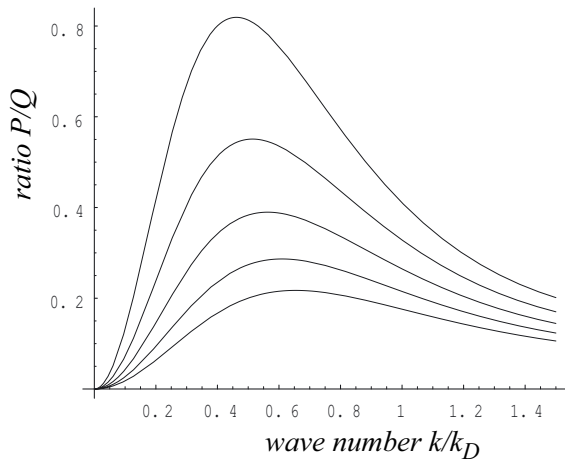
Fig. 2. The ratio $\eta = P/Q$ of the coefficients in the NLSE (22) is depicted versus the wave number k (normalized over k_D), for several values of the dust parameter μ; in descending order (from top to bottom): 0.8, 0.9, 1.0, 1.1, 1.2.

which also represents a localized region of negative wave density. Comparing to the dark soliton (25), we note that the maximum amplitude ρ_2 is now finite (non-zero) everywhere; see Figure 1d. The the pulse width of this gray-type excitation $L'' = \sqrt{2|P/Q|}/(a\,\rho_2)$ now also depends on an independent parameter a which represents the modulation depth ($0 < a \leq 1$). The lengthy expressions which determine the phase shift Θ and the parameter a, which are omitted here for brevity, can be found in references [5,16]. For $a = 1$, one recovers the *dark* soliton presented above.

An important qualitative result to be retained is that the envelope soliton width L and maximum amplitude ρ satisfy $L\rho \sim \sqrt{P/Q}$ (see above), and thus depend on (the ratio of) the coefficients P and Q; for instance, regions with higher values of P (or lower values of Q) will support wider (spatially more extended) localized excitations, for a given value of the maximum amplitude. Contrary to the KdV soliton picture, the width of these excitations does not depend on their velocity. It does, however, depend on the plasma parameters, e.g. here μ.

The localized envelope excitations presented above represent the slowly varying envelope which confines the (fast) carrier space and time oscillations, viz. $\phi = \Psi(X, Z)\cos(k\xi - \omega\tau)$ for the electric potential ϕ (and analogous expressions for the density n_i etc.; cf. (21)). The qualitative characteristics (width, amplitude) of these excitations, may be investigated by a numerical study of the ratio $P/Q \equiv \eta(k; \mu)$: recall that its sign determines the type (bright or dark) of the excitation, while its (absolute) value determines its width for a given amplitude (and vice versa). In Figure 2 we have depicted the behaviour of η as a function of the wavenumber k and the parameter μ: higher values of μ correspond to lower curves. Remember that, for any given wavenumber k, the dust concentration (expressed via the value of μ) determines the soliton width L (for a given amplitude ρ; see discussion above) since $L \sim \eta^{1/2}/\rho$. Therefore, we see that the addition of

negative dust generally ($\mu < 1$) results to higher values of η (i.e. wider or higher solitons), while *positive* dust ($\mu > 1$) has the opposite effect: it reduces the value of η (leading to narrower or shorter solitons). In a rather general manner, bright type solitons (pulses) seem to be rather privileged, since the ratio η (or the product PQ) of the coefficients P and Q is positive in most of the k, μ plane of values. One exception seems to be very the region of *very* low values of μ (typically below 0.2), which develops a negative tail of η for small k ($<0.3\,k_D$): thus, a very high (>80 per cent) electron depletion results in pulse destabilization in favour of dark-type excitations (Figs. 1c and 1d). Strictly speaking, η also becomes negative for very high wave number values ($>2.5\,k_D$); nevertheless, we neglect — for rigor — this region from the analysis, in this (long wavelength λ) fluid picture (for a weak dust presence, short λ DIAWs may be quite strongly damped; however, this result may still be interesting for a strong presence of dust, when Landau damping is not a significant issue [1]).

5 Relation to previous works: an approximate nonlinear Lagrangian treatment

By combining the Lagrangian system of equations (12–16), one obtains the (reduced) evolution equation

$$\frac{\partial^2}{\partial\tau^2}\left(\frac{1}{n}\right) = -\frac{1}{n_0}\frac{\partial}{\partial\xi}\left[\frac{n}{n_0}\frac{\partial}{\partial\xi}\,g(w)\right], \qquad (27)$$

where the function $g(x)$ [defined in Eq. (8)] is evaluated at

$$w(n) = n\left[1 - \delta^2\frac{\partial^2}{\partial\tau^2}\left(\frac{1}{n}\right)\right].$$

Note that the ion density n has been scaled by its equilibrium value $n_{i,0}$, to be distinguished from the initial condition $n_0 = n(\xi, \tau = 0)$.

Despite its complex form, the nonlinear evolution equation (27) can be solved exactly by considering different special cases, as regards the order of magnitude of the dispersion–related parameter δ. This treatment, based on reference [9], will only be briefly summarized here, for the sake of reference.

First, one may consider very short scale variations, i.e. $L \ll \lambda_D$ (or $\delta \gg 1$). This amounts to neglecting collective effects, so oscillatory motion within a Debye sphere is essentially decoupled from neighboring ones. By considering $w(n) \approx -\delta^2 n\,\partial^2(1/n)/\partial\tau^2$ and $\phi \approx 0$ (i.e. $\hat{n} \approx 1$), one may combine equations (15) and (27) into

$$\left(\frac{\partial^2}{\partial\tau^2} + \omega_{p,i}^2\right)\left(\frac{1}{n} - 1\right) = 0, \qquad (28)$$

which, imposing the initial condition $n(\xi, 0) = n_0(\xi)$, yields the solution

$$n(\xi, \tau) = \frac{n_0(\xi)}{\frac{n_0(\xi)}{n_{i,0}} + \left(1 - \frac{n_0(\xi)}{n_{i,0}}\right)\cos\omega_{p,i}\tau}. \qquad (29)$$

Note that if the system is initially at equilibrium, viz. $n_0(\xi) = n_{i,0}$, then it remains so at all times $\tau > 0$. Now, one may go back to equation (12) and solve for $\alpha(\xi, \tau)$, which in turn immediately provides the mean fluid velocity u

$$u(\xi, \tau) = \omega_{p,i} \sin \omega_{p,i} \tau \int_{\xi_0}^{\xi} \left(1 - \frac{n_0(\xi')}{n_{i,0}} \right) d\xi'$$

via (10), and then $E(\xi, \tau)$ and $\phi(\xi, \tau)$. Finally, the variable transformation (9) may now be inverted, immediately providing the Eulerian position x in terms of ξ and τ. We shall not go into further details regarding this procedure, which is essentially analogue (yet not identical) to Davidson's treatment of electron plasma oscillations.

Quite interestingly, upon neglecting the dispersive effects, i.e. setting $\delta = 0$, equation (27) may be solved by separation of variables, and thus shown to possess a nonlinear special solution in the form of a product, say $n(\xi, \tau) = n_1(\xi)n_2(\tau)$ [13]. This calculation was put forward in reference [9] (where the study of IAW — in a single electron temperature plasma — was argued to rely on an equation quasi-identical to Eq. (27)). However, the solution thus obtained relies on doubtful physical grounds, since the assumption $\delta \approx 0$, which amounts to remaining close to equilibrium — cf. the last of equations (6), implies an infinite space scale L (recall the definition of δ), contrary to the very nature of the (localized) nonlinear excitation itself. Rather not surprisingly, this solution was shown in reference [9] to decay fast in time, in both Eulerian and Lagrangian coordinates. Therefore, we shall not pursue this analysis any further.

6 Discussion and conclusions

We have studied the nonlinear propagation of dust ion acoustic waves propagating in a dusty plasma. By employing a Lagrangian formalism, we have investigated the modulational stability of the amplitude of the propagating dust ion acoustic oscillations and have shown that these electrostatic waves may become unstable, due to self interaction of the carrier wave. This instability may either lead to wave collapse or to wave energy localization, in the form of propagating localized envelope structures. We have provided an exact set of analytical expressions for these localized excitations.

This study complements similar investigations which relied on an Eulerian formulation of the dusty plasma fluid model [5]. In fact, the Lagrangian picture provides a strongly modified nonlinear stability profile for the wave amplitude, with respect to the previous (Eulerian) description; this was intuitively expected, since the passing to Lagrangian variables involves an inherently nonlinear transformation, which inevitably modifies the nonlinear evolution profile of the system described. However, the

general qualitative result remains in tact: the dust ion acoustic-type electrostatic plasma waves may propagate in the form of localized envelope excitations, which are formed as a result of the mutual balance between dispersion and nonlinearity in the plasma fluid. More sophisticated descriptions, incorporating e.g. thermal or collisional effects, may be elaborated in order to refine the parameter range of the problem, and may be reported later.

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